Time-dependent singular differential equations

Xavier Gràcia and Rubén Martín

Departament de Matemàtica Aplicada IV
Universitat Politècnica de Catalunya
Campus Nord UPC, edifici C3
C/ Jordi Girona 1, 08034 Barcelona
Catalonia, Spain

emails: xgracia@mat.upc.es, rubenmg@mat.upc.es

25 October 2004

Abstract

A geometric framework for describing and solving time-dependent implicit differential equations $F(t, x, x') = 0$ is studied, paying special attention to the linearly singular case, where F is affine in the velocities: $A(t, x)x' = b(t, x)$. This framework is based on the jet bundle of a time-dependent configuration space, and is an extension of the geometric framework of the autonomous case. When A is a singular matrix, the solutions can be obtained by means of constraint algorithms, either directly or through an equivalent autonomous system that can be constructed using the vector hull functor of affine spaces. As applications, we consider the jet bundle description of time-dependent lagrangian systems and the Skinner–Rusk formulation of time-dependent mechanics.

1 Introduction

It is known that the dynamics of an autonomous mechanical system given by a lagrangian function $L\colon \mathrm{T}Q \to \mathbf{R}$ is described by equations of motion of the type $W(q, \dot{q})\, \ddot{q} = \alpha(q, \dot{q})$; writing them as a first-order differential equation on $\mathrm{T}Q$, we obtain

$$A(x)\dot{x} = b(x),$$

where the matrix A is singular if the lagrangian is not regular (see for instance [Car 90]). Indeed, such differential equations appear in many other applications as control theory, circuit theory, engineering, ... —see some references in [GMR 04].

A general framework to deal with differential equations of this type was given in [GP 91] and developed in [GP 92, GP 02]: the so-called *linearly singular systems*. Since the matrix A may be singular, the system may not have solutions passing through each point of $\mathrm{T}Q$, and the solutions may not be unique. To solve this problem, a constraint algorithm has to be performed. If the algorithm ends, one obtains a submanifold of $\mathrm{T}Q$ where there exist solutions of the system. In fact, this algorithm is a generalization of the presymplectic constraint algorithm [GNH 78], which is in its turn a geometric version of the Dirac–Bergmann theory of constrained systems.

The main purpose of this paper is to extend these results to the time-dependent case. We study the geometric framework of time-dependent first-order implicit differential equations,

$$F(t, x, \dot{x}) = 0,$$

and the linearly singular case, when F is affine in the velocities,

$$A(t, x)\dot{x} = b(t, x),$$

where A is a possibly singular matrix.

Our model for the time-dependent configuration space, rather than a trivial product $M = \mathbf{R} \times Q$, is a fibre bundle [Sau 89] $\rho\colon M \to \mathbf{R}$, where the base $\mathbf{R}$ contains the time variable. Such an M is isomorphic to a product $\mathbf{R} \times Q$, but in practical applications there may not be a privileged trivialization, and a possible extension to deal with field theory of course should not be based on a trivial bundle. Some references about time-dependent lagrangian systems are [EMR 91, CF 93, CLM 94] in the product case and [Kru 97, MPL 03, MS 98, LMMMR 02] in the fibre bundle case; see also references therein. Time-dependent systems in general are studied in many books, as for instance [AM 78, Olv 93].

The main difference between the autonomous and the non-autonomous case is the usage of tangent bundles and jet bundles respectively. To describe an autonomous differential equation on the configuration space Q we use the tangent bundle $\mathrm{T}Q$, which is a vector bundle. To describe a non-autonomous differential equation on the time-dependent configuration space M, we use its jet bundle $\mathrm{J}^1\rho$, which is an affine bundle over M. To describe a linearly singular equation on M we will use affine morphisms defined on this affine bundle.

We will also propose a constraint algorithm for time-dependent singular systems. This algorithm is the natural generalization of the algorithm for the autonomous case —both in

the general implicit [RR 94, MMT 95] and linearly singular cases [GP 91, GP 92]. The case of an implicit equation in a product $M = \mathbf{R} \times Q$ has already been discussed in [Del 04]. It is worth noting that constraint algorithms for some time-dependent systems have been described in several papers, as for instance [CF 93, CLM 94, ILMM 99, LMM 96, LMMMR 02, Vig 00]. All these systems are of linearly singular type, so they are included within our framework.

When studying a time-dependent differential equation, sometimes it is useful to convert it into an equivalent time-independent one. This is even more interesting for implicit equations; for instance, the constraint algorithm for the autonomous case is easier to implement than for the non-autonomous case, because of the fact that vector fields instead of jet fields are used to obtain the constraint functions.

Therefore, we will examine the possibility of associating an autonomous linearly singular system with a time-dependent one, so that the solutions of both systems will be in correspondence. Essentially, we use the canonical inclusion of $\mathrm{J}^1\rho$ into $\mathrm{T}M$. In order to perform this association, we propose two different strategies. One possibility is to choose a connection on the jet bundle to induce a splitting of the tangent bundle. The other possibility, which does not make use of any choice, is based on the notion of vector hull of an affine space or an affine bundle. The main idea is that any affine space A can be canonically embedded as a hyperplane in a vector space $\hat{A}$ —the vector hull of A; with this immersion affine maps can be homogenized, that is, converted into linear maps.

As we have pointed out at the beginning, our main motivation comes from Euler–Lagrange equations and mechanical systems, where equations of motion are of second order. So it is interesting to extend the preceding study to second-order implicit and linearly singular equations:

$$F(t, x, \dot{x}, \ddot{x}) = 0, \qquad A(t, x, \dot{x})\ddot{x} = b(t, x, \dot{x}).$$

As applications of the formalism, we give two descriptions of time-dependent mechanical systems, in the form of time-dependent singular lagrangian systems and in the mixed velocity-momentum description (sometimes called Skinner–Rusk formulation [Ski 83, SR 83]) of time-dependent mechanics [CMC 02]. As a concrete example, we also study a pendulum of variable length.

The paper is organized as follows. In section 2 we study the geometric formulation of time-dependent differential equations, either in the implicit and in the linearly singular case. In section 3 we describe constraint algorithms for both cases. In section 4 we present two constructions of an autonomous system associated with a given time-dependent system. The extension to second-order equations is presented in section 5. Applications to singular lagrangian mechanics are presented in section 6, and section 7 is devoted to an example. Finally, there is an appendix about vector hulls of affine spaces and affine bundles.

2 Time-dependent systems

In this section we discuss first-order time-dependent singular differential equations. As a model of time-dependent configuration space, we take a fibre bundle $\rho\colon M \to \mathbf{R}$ over the real line (though more general settings could be also considered).

The appropriate geometric framework to deal with derivatives is that of jet bundles, so we will begin by giving some facts and notation about them —see for instance [Sau 89].

2.1 First-order jet bundles

We denote by $\mathrm{J}^1\rho$ the first-order jet manifold of ρ. Its elements, called 1-jets, are equivalence classes of local sections of ρ: two sections are equivalent at a point when they are tangent. We denote by $\mathrm{j}_t^1\xi$ the 1-jet of a section ξ at t.

$\mathrm{J}^1\rho$ is a fibre bundle over M and over $\mathbf{R}$, with canonical projections shown in the diagram:

$$
\begin{array}{ccc}
 & \mathrm{J}^1\rho & \\
\rho_{1,0} \downarrow & & \searrow \rho_1 \\
M & \xrightarrow{\ \rho\ } & \mathbf{R}
\end{array}
$$

We denote by t the canonical coordinate of $\mathbf{R}$. If (t, q^i) are fibred coordinates on M, then $\mathrm{J}^1\rho$ has induced coordinates (t, q^i, v^i).

The bundle $\rho_{1,0}\colon \mathrm{J}^1\rho \to M$ is an affine bundle modelled on the vertical bundle of ρ, $\mathrm{V}\rho \to M$. Recall that the vertical bundle $\mathrm{V}\rho$ is a subbundle of $\mathrm{T}M$, given by $\mathrm{V}\rho = \mathrm{Ker}\,\mathrm{T}\rho$. The elements of $\mathrm{V}\rho$ are the tangent vectors of M which are tangent to the fibres; equivalently, $\mathrm{V}\rho = \{v \in \mathrm{T}M \mid i_v\mathrm{d}t = 0\}$.

There is a canonical embedding $\iota\colon \mathrm{J}^1\rho \to \mathrm{T}M$, defined as $\iota(\mathrm{j}_t^1\xi) = \dot{\xi}(t)$. In local coordinates, $\iota(t, q^i, v^i) = (t, q^i, 1, v^i)$. Notice that $\iota(\mathrm{J}^1\rho) = \{v \in \mathrm{T}M \mid i_v\mathrm{d}t = 1\}$.

2.2 Implicit systems

In general, a (time-dependent) *implicit differential equation* is defined by a submanifold $\mathcal{D} \subset \mathrm{J}^1\rho$. A local section $\xi\colon I \to M$ of ρ is a *solution* of the differential equation if

$$\mathrm{j}^1\xi(t) \in \mathcal{D} \tag{1}$$

for each t. If the subset $\mathcal{D}$ is locally described in coordinates by some equations $F^\alpha(t, q^i, v^i) = 0$, then the differential equation reads $F^\alpha(t, \xi^i(t), \dot{\xi}^i(t)) = 0$.

Suppose that $\mathcal{D}$ is the image of a jet field, that is, of a section $X\colon M \to \mathrm{J}^1\rho$. Then the solutions of the differential equation are the integral sections of X, which are the solutions of the explicit differential equation

$$\mathrm{j}^1\xi = X \circ \xi.$$

In coordinates, if $X(t, q^i) = (t, q^i, X^j(t, q^i))$, the differential equation reads $\dot{\xi}^i(t) = X^i(t, \xi^j(t))$.

Consider again $\mathcal{D} \subset \mathrm{J}^1\rho$. Given a jet field X, its integral sections are solutions of the implicit equation defined by $\mathcal{D}$ iff

$$X(M) \subset \mathcal{D}. \tag{2}$$

So, in a certain sense, solving this equation is equivalent to solving the implicit equation (1).

For an explicit differential equation there always exist solutions, and each initial condition $x \in M$ defines a unique maximal solution. For an implicit differential equation existence and uniqueness may fail; in this case one is lead to study the subset of points covered by solutions, and the multiplicity of solutions.

2.3 Linearly singular systems

A (time-dependent) *linearly singular system* on M is defined by a vector bundle $\pi\colon E \to M$ and an affine bundle morphism $\mathcal{A}\colon \mathrm{J}^1\rho \to E$:

$$\begin{array}{ccc} \mathrm{J}^1\rho & \xrightarrow{\;\mathcal{A}\;} & E \\ {\scriptstyle \rho_{1,0}}\downarrow & \swarrow{\scriptstyle \pi} & \\ M & & \end{array} \tag{3}$$

For the sake of brevity, we will refer to this linearly singular system simply as $\mathcal{A}$.

The system $\mathcal{A}$ has an associated implicit system given by

$$\mathcal{D} = \mathcal{A}^{-1}(0) \subset \mathrm{J}^1\rho. \tag{4}$$

A local section $\xi\colon I \to M$ is a solution of $\mathcal{D}$, equation (1), iff it is a solution of the linearly singular differential equation

$$\mathcal{A} \circ \mathrm{j}^1\xi = 0. \tag{5}$$

In local coordinates, the bundle morphisms are given by

$$\pi(t, q^i, u^\alpha) = (t, q^i), \qquad \mathcal{A}(t, q^i, v^i) = (t, q^i, \mathcal{A}_j^\alpha(t, q^i)v^j + c^\alpha(t, q^i)),$$

thus the differential equation reads

$$\mathcal{A}_j^\alpha(t, \xi^i(t))\,\dot{\xi}^j(t) + c^\alpha(t, \xi^i(t)) = 0.$$

As before, it may be convenient to describe the solutions of the differential equation as integral sections of jet fields. A jet field $X\colon M \to \mathrm{J}^1\rho$ is a solution jet field of $\mathcal{D}$, equation (2), iff

$$\mathcal{A} \circ X = 0. \tag{6}$$

Then its integral sections are solutions of the differential equation defined by $\mathcal{A}$.

Locally, $X(t, q^i) = (t, q^i, X^i(t, q^i))$ is a solution jet field of $\mathcal{A}$ when

$$\mathcal{A}_j^\alpha(t, q^i)X^j(t, q^i) + c^\alpha(t, q^i) = 0.$$

Let us remark that, instead of a vector bundle, we could have considered an affine bundle E with a section b and an affine bundle morphism $\mathcal{A}\colon \mathrm{J}^1\rho \to E$. This slight generalization does not seem too relevant for applications, and indeed the section b endows E with a vector bundle structure.

3 Constraint algorithm

In general, an implicit system does not have solution jet fields, and does not have solution sections passing through every point in M. We want to find a maximal subbundle $\rho'\colon M' \to \mathbf{R}$ of ρ (over $\mathbf{R}$ for simplicity, but more general situations could occur) where there exist solution jet fields $X\colon M' \to \mathrm{J}^1\rho'$ and solution sections $\xi\colon I \to M'$ through every point in M'.

To this end, we can adapt the constraint algorithms of the time-independent case, both for implicit systems [RR 94] [MMT 95] and linearly singular systems [GP 91, GP 92], to the time-dependent case. A constraint algorithm for a time-dependent implicit equation in a product $M = \mathbf{R} \times Q$ has been recently discussed in [Del 04].

3.1 Implicit systems

Let $\mathcal{D} \subset J^1\rho$ be an implicit system. We say that a 1-jet $y \in \mathcal{D}$ is *integrable* (or *locally solvable*) if there exists a solution $\xi: I \to M$ of $\mathcal{D}$ such that $j^1\xi$ passes through y. One of the purposes of the constraint algorithm is to find the set $\mathcal{D}_{\text{int}}$ of all integrable 1-jets.

If a solution passes through a point $x \in M$, necessarily x belongs to the subset

$$M_{(1)} := \rho_{1,0}(\mathcal{D}). \tag{7}$$

Denote by $\rho_{(1)}: M_{(1)} \to \mathbf{R}$ the restriction of ρ to $M_{(1)}$. To proceed with the algorithm, we will assume that $\rho_{(1)}$ is a subbundle of ρ. In this case, the inclusion $i_{(1)}: M_{(1)} \hookrightarrow M$ has a 1-jet prolongation, $j^1 i_{(1)}: J^1\rho_{(1)} \hookrightarrow J^1\rho$. By means of this inclusion, we can define

$$\mathcal{D}_{(1)} := J^1\rho_{(1)} \cap \mathcal{D}. \tag{8}$$

Since the solutions of $\mathcal{D}$ lay on $M_{(1)}$, the integrable jets of $\mathcal{D}$ must be contained in $\mathcal{D}_{(1)}$. If this is a submanifold, we have obtained a new implicit system, now on $M_{(1)}$.

This procedure can be iterated: from $M_{(0)} = M$ and $\mathcal{D}_{(0)} = \mathcal{D}$, and assuming that at each step one obtains subbundles and submanifolds, one may define $M_{(i)} := \rho_{1,0}(\mathcal{D}_{(i-1)})$ and $\mathcal{D}_{(i)} := J^1\rho_{(i)} \cap \mathcal{D}_{(i-1)}$. The algorithm finishes when, for some k, we have $M_{(k+1)} = M_{(k)}$. In this case, since $\rho_{1,0}(\mathcal{D}_{(k)}) = M_{(k)}$, if we suppose for instance that the projection $\mathcal{D}_{(k)} \to M_{(k)}$ is a submersion, we have that $\mathcal{D}_{\text{int}} = \mathcal{D}_{(k)}$.

3.2 Linearly singular systems

Let $\mathcal{A}: J^1\rho \to E$ be a time-dependent linearly singular system as described in section 2. We can proceed by applying the preceding algorithm for implicit systems, and also by adapting the algorithm for time-independent linearly singular systems.

So we begin with $\mathcal{D} = \mathcal{A}^{-1}(0)$. As before, the configuration space must be restricted to $M_{(1)} := \rho_{1,0}(\mathcal{D})$, which can also be described as

$$M_{(1)} = \{x \in M \mid 0_x \in \operatorname{Im} \mathcal{A}_x\};$$

note that $0_x \in \operatorname{Im} \mathcal{A}_x$ is the necessary consistency condition for (6) to hold on a given point $x \in M$.

As above, we assume that $\rho_{(1)}: M_{(1)} \to \mathbf{R}$ is a subbundle of M.

Let us restrict all the data to $M_{(1)}$: $\mathcal{A}_{(1)} := \mathcal{A}|_{J^1\rho_{(1)}}$, $E_{(1)} := E|_{M_{(1)}}$, and $\pi_{(1)} := \pi|_{M_{(1)}}$. So we obtain a linearly singular system on $M_{(1)}$:

$$\begin{array}{ccc}
J^1\rho_{(1)} & \xrightarrow{\mathcal{A}_{(1)}} & E_{(1)} \\
{\scriptstyle (\rho_{(1)})_{1,0}} \downarrow & \swarrow {\scriptstyle \pi_{(1)}} & \\
M_{(1)} & &
\end{array} \tag{9}$$

It is clear that the implicit system defined by $\mathcal{A}_{(1)}$ coincides with the implicit system $\mathcal{D}_{(1)}$ obtained above, (8), that is,

$$\mathcal{A}_{(1)}^{-1}(0) = J^1\rho_{(1)} \cap \mathcal{D}. \tag{10}$$

Thus, if we assume for instance that each $M_{(i)}$ is a subbundle of $M_{(i-1)}$, we obtain a constraint algorithm for the linearly singular case:

$$M_{(i)} := \{x \in M_{(i-1)} \mid 0_x \in \mathrm{Im}(\mathcal{A}_{(i-1)})_x\},$$

$$\mathcal{A}_{(i)} := \mathcal{A}_{(i-1)}\Big|_{\mathrm{J}^1\rho_{(i)}},$$

$$E_{(i)} := E_{(i-1)}\Big|_{M_{(i)}},$$

$$\pi_{(i)} := \pi_{(i-1)}\Big|_{M_{(i)}}.$$

When the algorithm finishes, we arrive to a final system which is integrable everywhere.

$$
\begin{array}{ccc}
\mathrm{J}^1\rho_{(k)} & \xrightarrow{\ \mathcal{A}_{(k)}\ } & E_{(k)} \\
{\scriptstyle (\rho_{(k)})_{1,0}}\Big\downarrow & \swarrow{\scriptstyle \pi_{(k)}} & \\
M_{(k)} & &
\end{array}
$$

4 From non-autonomous to autonomous systems

It is usual to convert a time-dependent system into a time-independent one by considering the evolution parameter as an new dependent variable. From a geometric viewpoint, this is easily done with an implicit system $D \subset \mathrm{J}^1\rho$ by means of the canonical inclusion $\iota \colon \mathrm{J}^1\rho \to \mathrm{T}M$: its image $E = \iota(D)$ is a submanifold of $\mathrm{T}M$, so it defines an autonomous implicit equation. The equivalence between both equations is immediate:

Proposition 1 *A map $\xi \colon \mathbf{R} \to M$ is a solution section of the time-dependent system D iff it is a solution path of the autonomous system E such that $\rho(\xi(t_0)) = t_0$ for any arbitrarily given t_0.*∎

Now let us focus on the case of a time-dependent linearly singular system $\mathcal{A}$:

$$
\begin{array}{ccc}
\mathrm{J}^1\rho & \xrightarrow{\ \mathcal{A}\ } & E \\
{\scriptstyle \rho_{1,0}}\Big\downarrow & \swarrow{\scriptstyle \pi} & \\
M & &
\end{array}
\tag{11}
$$

We will also relate this system to an autonomous one. The main motivation for finding such a relation is that the constraint algorithm described in the preceding section is easier to implement in the autonomous case. The reason is that, instead of jet fields, vector fields can be used to obtain constraint functions that define the submanifolds in the constraint algorithm, as will be shown later on.

Two constructions to achieve our goal will be proposed. In the first one, we use a connection to define a complement of $\mathrm{V}\rho$ in $\mathrm{T}M$. In the second construction, we use the vector hull functor described in the appendix to define vector bundles and morphisms from affine bundles and morphisms.

Previously, we shall recall some definitions concerning the autonomous case [GP 92]. An *autonomous linearly singular system* on a manifold N is defined by a vector bundle $\pi\colon F \to N$, a vector bundle morphism $A\colon \mathrm{T}N \to F$, and a section $b\colon N \to F$:

$$\begin{array}{ccc} \mathrm{T}N & \xrightarrow{\ A\ } & F \\ {\scriptstyle \tau_N}\Big\downarrow & {\scriptstyle \pi} \nearrow {\scriptstyle b} \nearrow & \\ N & & \end{array}$$

Taking local coordinates $(x^i, \dot{x}^i)$ on $\mathrm{T}N$ and (x^i, u^α) on F, the local expressions of the maps are

$$\pi(x^i, u^\alpha) = (x^i), \qquad b(x^i) = (x^i, b^\alpha(x^i)), \qquad A(x^i, \dot{x}^i) = (x^i, A_j^\alpha(x^i)\dot{x}^j).$$

We say that a path $\gamma\colon I \to N$ is a *solution path* if

$$A \circ \dot{\gamma} = b \circ \gamma.$$

Locally, $A_j^\alpha(\gamma(t))\dot{\gamma}^j(t) = b^\alpha(\gamma(t))$. A vector field $X \in \mathfrak{X}(N)$ is a *solution vector field* when

$$A \circ X = b;$$

in coordinates, $A_j^\alpha(x)X^j(x) = b^\alpha(x)$.

Let us roughly recall —see [GP 92] for a detailed description— how the constraint algorithm is explicitly carried out in this autonomous case, that is, how the consecutive constraint manifolds are effectively computed. First of all, it can be seen that the primary constraint submanifold $M_1 = \{x \in M \mid b(x) \in \mathrm{Im}\, A_x\}$ is locally described by the vanishing of the functions $\phi^\alpha := \langle s^\alpha, b\rangle$, where (s^α) is a local frame for $\mathrm{Ker}\,{}^tA \subset F^*$. The constancy of the rank of A is needed here to ensure that this procedure works.

We have that a vector field X in M, in order to be a solution of the system, must satisfy the equation $A \circ X \underset{M_1}{\simeq} b$. Vector fields satisfying this condition always exist and are called primary vector fields. Given one primary vector field X_0, the others have the form $X \underset{M_1}{\simeq} X_0 + \sum_\mu f^\mu \Gamma_\mu$, where f^μ are functions uniquely determined on M_1 and (Γ_μ) is a local frame for $\mathrm{Ker}\,A$.

A primary vector field X can be a solution of the system only if it is tangent to M_1, so we obtain the equation, for every constraint ϕ^α, $(X \cdot \phi^\alpha)(x) = 0$, for $x \in M_1$, or, equivalently, $(X_0 \cdot \phi^\alpha)(x) + \sum_\mu (\Gamma_\mu \cdot \phi^\alpha)(x)f^\mu(x) = 0$. These equations may provide new constraints that define the secondary constraint submanifold M_2, and may also determine some of the functions f^μ.

This procedure can be iterated until we determine which primary vector fields are solutions of the system, and we obtain the final submanifold where they are defined.

4.1 Jet field construction

Consider the linearly singular system given by (11). Let us choose an arbitrary jet field $\Gamma\colon M \to \mathrm{J}^1\rho$. This jet field Γ induces [Sau 89] in a natural way a connection $\tilde{\Gamma}$ on the bundle ρ and a splitting $\mathrm{T}M = \mathrm{V}\rho \oplus H_\Gamma$ of the tangent bundle, with projections v_Γ and h_Γ. The coordinate expressions are:

$$\Gamma(t, q^i) = (t, q^i, \Gamma^i(t, q^i)),$$

$$\tilde{\Gamma} = \mathrm{d}t \otimes \left(\frac{\partial}{\partial t} + \Gamma^i \frac{\partial}{\partial q^i} \right),$$

$$v_\Gamma(t, q^i; \dot{t}, \dot{q}^i) = (t, q^i; 0, \dot{q}^i - \dot{t}\,\Gamma^i(t, q^i)), \qquad h_\Gamma(t, q^i; \dot{t}, \dot{q}^i) = (t, q^i; \dot{t}, \dot{t}\,\Gamma^i(t, q^i)).$$

From this we can define a section of π,

$$b_\Gamma := -\mathcal{A} \circ \Gamma \colon M \to E,$$

and a vector bundle morphism,

$$A_\Gamma := \vec{\mathcal{A}} \circ v_\Gamma \colon \mathrm{T}M \to E,$$

where $\vec{\mathcal{A}} \colon \mathrm{V}\rho \to E$ is the vector bundle morphism associated with the affine map $\mathcal{A}$.

With these objects we can construct a time-independent linearly singular system:

$$\begin{array}{ccc} \mathrm{T}M & \xrightarrow{\ A_\Gamma \oplus \mathrm{d}t\ } & E \oplus \mathbf{R} \\ {\scriptstyle \tau_M}\downarrow & \nearrow {\scriptstyle b_\Gamma \oplus 1} & \\ M & & \end{array} \tag{12}$$

This system is equivalent to the time-dependent system (11) in the sense of the following proposition.

Proposition 2 *Consider the time-dependent system given by (11). Given any jet field $\Gamma\colon M \to \mathrm{J}^1\rho$, we have:*

i) A map $\xi\colon \mathbf{R} \to M$ is a solution section of the time-dependent system (11) if, and only if, it is a solution path of the autonomous system (12) such that $\rho(\xi(t_0)) = t_0$ for any arbitrarily given t_0.

ii) A map $X\colon M \to \mathrm{J}^1\rho$ is a solution jet field of the time-dependent system (11) if, and only if, considered as a vector field in M, it is a solution vector field of the autonomous system (12).

(Note that we are using the embedding $\iota\colon \mathrm{J}^1\rho \to \mathrm{T}M$ defined in section 2 to identify jet fields as vector fields.)

Proof. It is immediate, taking into account the local expressions of the equations defined respectively by both systems:

- $\mathcal{A}_j^\alpha(t, q^i)\, v^j + c^\alpha(t, q^i) = 0$

- $\begin{cases} \mathcal{A}_j^\alpha(t, q^i)\,(\dot{q}^j - \dot{t}\,\Gamma^j(t, q^i)) = -\mathcal{A}_j^\alpha(t, q^i)\,\Gamma^j(t, q^i) - c^\alpha(t, q^i) \\ \dot{t} = 1 \end{cases}$ $\blacksquare$

4.2 Vector hull construction

Here we will apply the vector hull functor described in the appendix. The affine bundle morphism $\mathcal{A}$ in (11) induces a vector bundle morphism $\widehat{\mathcal{A}}$ between the vector hulls of $\mathrm{J}^1\rho$ and E:

$$
\begin{array}{ccc}
\mathrm{J}^1\rho & \hookrightarrow & \widehat{\mathrm{J}^1\rho} \\
\mathcal{A} \downarrow & & \downarrow \widehat{\mathcal{A}} \\
E & \underset{i}{\hookrightarrow} & \widehat{E}
\end{array}
$$

The 0 section of $\pi\colon E \to M$ also induces a section $\hat{0}$ of $\widehat{\pi}\colon \widehat{E} \to M$, defined by $\hat{0} = i \circ 0$. Recall that with the identification $\widehat{E} = E \times \mathbf{R}$, we have $\hat{0} = (0, 1)$.

Using the canonical identification of $\widehat{\mathrm{J}^1\rho}$ with $\mathrm{T}M$, we can construct the following linearly singular system:

$$
\begin{array}{ccc}
\mathrm{T}M & \xrightarrow{\widehat{\mathcal{A}}} & \widehat{E} \\
\tau_M \downarrow & \nearrow{\hat{0}} & \\
M & &
\end{array}
\tag{13}
$$

This system is equivalent to the time-dependent system (11) in the sense of the following proposition.

Proposition 3 *Consider the time-dependent system given by (11).*

    i) A map $\xi\colon \mathbf{R} \to M$ is a solution section of the time-dependent system (11) if, and only if, it is a solution path of the associated autonomous system (13) such that $\rho(\xi(t_0)) = t_0$ for any arbitrarily given t_0.

    ii) A map $X\colon M \to \mathrm{J}^1\rho$ is a solution jet field of the time-dependent system (11) if, and only if, considered as a vector field in M, it is a solution vector field of the associated autonomous system (13).

Proof. Again we can prove the result in local coordinates, where the equations of the systems (11) and (13) read, respectively,

- $c^\alpha(t, q^i) + \mathcal{A}_j^\alpha(t, q^i)v^j = 0$

- $\begin{cases} c^\alpha(t, q^i)\dot{t} + \mathcal{A}_j^\alpha(t, q^i)\dot{q}^j = 0 \\ \dot{t} = 1 \end{cases}$

We have used the coordinates (u^0, u^α) on $\widehat{E}$ induced by the affine frame $(0; u^\alpha)$ on E. ∎

5 The second-order case

The preceding results can be extended to consider higher-order implicit and linearly singular differential equations. Of course, the most important case, due to its applications to mechanics, is that of second-order equations, to which we devote this section. Before proceeding, we need some additional facts about the second-order jet bundle of a fibre bundle $\rho\colon M \to \mathbf{R}$.

5.1 Second-order jet bundles

The jet space $\mathrm{J}^2\rho$ is a fibre bundle over $\mathrm{J}^1\rho$, M and $\mathbf{R}$. The canonical projections are:

The bundle $\rho_{2,1}\colon \mathrm{J}^2\rho \to \mathrm{J}^1\rho$ is an affine bundle modelled on the vertical bundle of $\rho_{1,0}$, $\mathrm{V}\rho_{1,0} \to \mathrm{J}^1\rho$.

There exists a natural embedding $\jmath\colon \mathrm{J}^2\rho \to \mathrm{J}^1\rho_1$, defined by $\jmath(\mathrm{j}_t^2\xi) = \mathrm{j}_t^1(\mathrm{j}^1\xi)$. In addition to $\jmath$, we can use the embedding $\iota\colon \mathrm{J}^1\rho_1 \to \mathrm{TJ}^1\rho$ to include the second-order jet bundle into a tangent bundle:

$$
\begin{array}{ccccc}
\mathrm{J}^2\rho & \xrightarrow{\ \jmath\ } & \mathrm{J}^1\rho_1 & \xrightarrow{\ \iota\ } & \mathrm{TJ}^1\rho \\
\mathrm{j}_t^2\xi & \longrightarrow & \mathrm{j}_t^1(\mathrm{j}^1\xi) & \longrightarrow & (\mathrm{j}^1\xi)^{\cdot}(t)
\end{array}
$$

Consider natural coordinates (t, q^i, v^i, a^i) on $\mathrm{J}^2\rho$. With them, the local expression of this composed embedding $\kappa = \iota \circ \jmath$ reads

$$
\kappa(t, q^i, v^i, a^i) = (t, q^i, v^i, 1, v^i, a^i).
$$

This shows that the image of $\mathrm{J}^2\rho$ by the embedding can be expressed as

$$
\kappa(\mathrm{J}^2\rho) = \{w \in \mathrm{TJ}^1\rho \mid i_w\mathrm{d}t = 1,\ S(w) = 0\}.
$$

Here there appears another relevant operator, the canonical vertical endomorphism S of $\mathrm{TJ}^1\rho$, whose local expression is

$$
S = (\mathrm{d}q^i - v^i\mathrm{d}t) \otimes \frac{\partial}{\partial v^i};
$$

note also that $\mathrm{Im}(S) = \mathrm{V}\rho_{1,0}$.

The Cartan distribution of $\mathrm{J}^1\rho$ is the kernel of the vertical endomorphism S of $\mathrm{T}(\mathrm{J}^1\rho)$. We denote it by $C\rho_{1,0}$. Locally, we can describe $C\rho_{1,0}$ as the distribution generated by the $n+1$ vector fields $\{\frac{\partial}{\partial t} + v^i\frac{\partial}{\partial q^i}, \frac{\partial}{\partial v^i}\}$. We have an exact sequence

$$
0 \longrightarrow C\rho_{1,0} \longrightarrow \mathrm{T}(\mathrm{J}^1\rho) \xrightarrow{\ S\ } \mathrm{V}\rho_{1,0} \longrightarrow 0 \tag{14}
$$

5.2 Second-order implicit and linearly singular systems

Similar to the first-order case, a *second-order implicit differential equation* is defined by a submanifold $\mathcal{D} \subset \mathrm{J}^2\rho$. A local section $\xi\colon I \to M$ is a *solution* of the differential equation if $\mathrm{j}^2\xi(t) \in \mathcal{D}$ for each t. In coordinates, this equation can be expressed as $F^\alpha(t, \xi^i(t), \dot\xi^i(t), \ddot\xi^i(t)) = 0$. Like in the first-order case, if $\mathcal{D}$ is the image of a section $X\colon \mathrm{J}^1\rho \to \mathrm{J}^2\rho$, the equation can be written in normal form.

Now let us consider the linearly singular case. A *time-dependent second-order linearly singular system* is defined by a vector bundle $\pi\colon E \to \mathrm{J}^1\rho$ and an affine bundle morphism $\mathcal{A}\colon \mathrm{J}^2\rho \to E$:

$$
\begin{array}{ccc}
\mathrm{J}^2\rho & \xrightarrow{\ \mathcal{A}\ } & E \\[4pt]
{\scriptstyle \rho_{2,1}}\downarrow & \swarrow{\scriptstyle \pi} & \\[4pt]
\mathrm{J}^1\rho & &
\end{array}
\tag{15}
$$

Its *solution sections* are sections ξ of ρ such that

$$
\mathcal{A} \circ \mathrm{j}^2\xi = 0.
$$

Locally this reads

$$
\mathcal{A}_j^\alpha(t, \xi^i(t), \dot\xi^i(t))\,\ddot\xi^j(t) + c^\alpha(t, \xi^i(t), \dot\xi^i(t)) = 0.
\tag{16}
$$

A second-order jet field, that is, a section X of $\rho_{(2,1)}\colon \mathrm{J}^2\rho \to \mathrm{J}^1\rho$, is a *solution jet field* if

$$
\mathcal{A} \circ X = 0;
$$

locally this reads $\mathcal{A}_j^\alpha(t, q^i, v^i)X^j(t, q^i, v^i) + c^\alpha(t, q^i, v^i) = 0$.

In a similar way of what we did in the previous section, it is interesting to convert the singular system given by (15) into a first-order autonomous linearly singular system. As before, we present two constructions of this.

5.3   Jet field construction

Choose an arbitrary second-order jet field $\Gamma\colon \mathrm{J}^1\rho \to \mathrm{J}^2\rho$. Again this determines a splitting of the tangent bundle of $\mathrm{J}^1\rho$ as a direct sum $\mathrm{T}\mathrm{J}^1\rho = \mathrm{V}\rho_{1,0} \oplus H_\Gamma$, with associated projections v_Γ and h_Γ. Now we define a section of π

$$
b_\Gamma := -\mathcal{A} \circ \Gamma\colon \mathrm{J}^1\rho \to E
$$

and a vector bundle morphism

$$
A_\Gamma := \vec{\mathcal{A}} \circ v_\Gamma\colon \mathrm{T}\mathrm{J}^1\rho \to E.
$$

With these definitions, we obtain an autonomous first-order linearly singular system on the manifold $\mathrm{J}^1\rho$:

$$
\begin{array}{ccc}
\mathrm{T}\mathrm{J}^1\rho & \xrightarrow{\ A_\Gamma \oplus S \oplus \mathrm{d}t\ } & E \oplus \mathrm{V}\rho_{1,0} \oplus \mathbf{R} \\[4pt]
\downarrow & \nearrow{\scriptstyle b_\Gamma \oplus 0 \oplus 1} & \\[4pt]
\mathrm{J}^1\rho & &
\end{array}
\tag{17}
$$

A result quite similar to Proposition 2 can be formulated, in the sense that this system is equivalent to the original time-dependent second-order system (15). This can be readily seen by comparing the local expression (16) with that of the equation defined by system (17). We skip the details.

5.4 Vector hull construction

As opposite to the first-order case, the vector hull of $\mathrm{J}^2\rho$ can not be identified with a tangent bundle, but rather with a tangent subbundle. As it is shown in the appendix, $\widehat{\mathrm{J}^2\rho}$ can be identified with the Cartan distribution $C\rho_{1,0}$ on $\mathrm{J}^1\rho$. Then, as in section 4.2, if we homogenize the system (15) we obtain the following:

$$
\begin{array}{cc}
C\rho_{1,0} \xrightarrow{\ \widehat{\mathcal{A}}\ } \widehat{E} & \qquad (18) \\
{\scriptstyle \tau_{\mathrm{J}^1\rho}}\Big\downarrow \ \ \nearrow{\scriptstyle \hat{0}} & \\
\mathrm{J}^1\rho &
\end{array}
$$

This is an autonomous linearly singular system on $\mathrm{J}^1\rho$, except for the fact that there is only a subbundle $C\rho_{1,0} \subset \mathrm{TJ}^1\rho$ instead of the whole tangent bundle. The interpretation of this system is the same as in the ordinary case, but with the additional requirement that, for a path $\eta\colon I \to \mathrm{J}^1\rho$, its derivative $\dot\eta$ must lie in $C\rho_{1,0}$ —which is a natural condition if η has to be the lift $\mathrm{j}^1\xi$ of a section of ρ. In coordinates, if $\eta = (t, q^i, v^i)$, this requirement amounts to

$$
\dot{q}^i = \dot{t}\, v^i.
$$

Assuming that $\dot\eta \subset C\rho_{1,0}$, the local equations for the path η to be a solution of the system (18) are

$$
\begin{cases}
\dot{t} = 1 \\
\dot{t}\, c^\alpha(t, q^i, v^i) + \dot{v}^j A_j^\alpha(t, q^i, v^i) = 0
\end{cases}
$$

Comparing these three equations with equation (16), we see that systems (15) and (18) are equivalent.

Finally, we will show that the system (18) can be expressed as a linearly singular system, provided we have an appropriate extension of $\mathcal{A}$. Since E is a vector bundle, we have a canonical identification $\widehat{E} = E \oplus \mathbf{R}$, and the vector extension $\widehat{\mathcal{A}}$ can be written $\widehat{\mathcal{A}} = \mathcal{A}\hat{} \oplus \mathrm{d}t$ (see the appendix). Suppose that we have an extension $\bar{A}\colon \mathrm{T}(\mathrm{J}^1\rho) \to E$ of the map $\mathcal{A}\hat{}\colon C\rho_{1,0} \to E$; in some applications (see next section) there exists a natural extension $\bar{A}$. Then the system (18) can be described as the following linearly singular system:

$$
\begin{array}{cc}
\mathrm{TJ}^1\rho \xrightarrow{\ \bar{A}\oplus \mathrm{d}t \oplus S\ } E \oplus \mathbf{R} \oplus \mathrm{V}\rho_{1,0} & \qquad (19) \\
\Big\downarrow \ \ \nearrow{\scriptstyle 0\oplus 1\oplus 0} & \\
\mathrm{J}^1\rho &
\end{array}
$$

The only thing to be noted is that $C\rho_{1,0}$ is the kernel of S, see (14).

6 Some applications to mechanics

6.1 Non-autonomous lagrangian systems

Let $\rho\colon Q \to \mathbf{R}$ be a fibre bundle modelling a time-dependent configuration space. A lagrangian system on Q is determined by a lagrangian function $L\colon \mathrm{J}^1\rho \to \mathbf{R}$.

The vertical endomorphism S of $\mathrm{T}(\mathrm{J}^1\rho)$ allows to construct the Poincaré–Cartan forms

$$\Theta_L = {}^tS \circ \mathrm{d}L + L\mathrm{d}t \ \in\ \Omega^1(\mathrm{J}^1\rho),$$

$$\Omega_L = -\mathrm{d}\Theta_L \ \in\ \Omega^2(\mathrm{J}^1\rho).$$

By contraction, this one defines a morphism $\hat{\Omega}_L \colon \mathrm{T}(\mathrm{J}^1\rho) \to \mathrm{T}^*(\mathrm{J}^1\rho)$.

In coordinates (t, q, v, a) of $\mathrm{J}^2\rho$, the Euler–Lagrange equation can be written $\mathrm{d}\hat{p}/\mathrm{d}t = \partial L/\partial q$, where $\hat{p} = \partial L/\partial v$. Now, given a vector field $X \in \mathfrak{X}(\mathrm{J}^1\rho)$, we can compute

$$i_X\Omega_L = \left((X{\cdot}q) - v(X{\cdot}t)\right)\mathrm{d}\hat{p} - \left((X{\cdot}\hat{p}) - \frac{\partial L}{\partial q}(X{\cdot}t)\right)\mathrm{d}q + \left((X{\cdot}\hat{p})v - \frac{\partial L}{\partial q}(X{\cdot}q)\right)\mathrm{d}t.$$

Consider the case where X is a second-order vector field, $X = \dfrac{\partial}{\partial t} + v\dfrac{\partial}{\partial q} + A(t,q,v)\dfrac{\partial}{\partial v}$, which amounts also to

$$i_X\mathrm{d}t = 1, \qquad S \circ X = 0. \tag{20}$$

Then the preceding expression simplifies to $i_X\Omega_L = \left((X{\cdot}\hat{p}) - \dfrac{\partial L}{\partial q}\right)(v\mathrm{d}t - \mathrm{d}q)$, and so the integral curves of X are a solution of the Euler–Lagrange equation iff

$$i_X\Omega_L = 0. \tag{21}$$

Now recall the affine inclusion $\kappa \colon \mathrm{J}^2\rho \hookrightarrow \mathrm{TJ}^1\rho$, which identifies jet fields $\mathrm{J}^1\rho \to \mathrm{J}^2\rho$ with second-order vector fields on $\mathrm{J}^1\rho$. From the preceding discussion, it is clear that the lagrangian dynamics may be described by the following second-order linearly singular system on Q:

$$
\begin{array}{ccc}
\mathrm{J}^2\rho & \xrightarrow{\ \hat{\Omega}_L \circ \kappa\ } & \mathrm{T}^*\mathrm{J}^1\rho \\
{\scriptstyle \rho_{2,1}}\Big\downarrow & \swarrow{\scriptstyle \tau^*_{\mathrm{J}^1\rho}} & \\
\mathrm{J}^1\rho & &
\end{array}
\tag{22}
$$

Using the vector hull construction described in the preceding section, we can convert this system into a first-order autonomous system on $\mathrm{J}^1\rho$:

$$
\begin{array}{ccc}
\mathrm{TJ}^1\rho & \xrightarrow{\ \hat{\Omega}_L \oplus \mathrm{d}t \oplus S\ } & \mathrm{T}^*\mathrm{J}^1\rho \oplus \mathbf{R} \oplus \mathrm{V}\rho_{1,0} \\
\Big\downarrow & \nearrow{\scriptstyle 0 \oplus 1 \oplus 0} & \\
\mathrm{J}^1\rho & &
\end{array}
$$

and note that its equations of motion are precisely (20), (21). If the lagrangian is regular, then this linearly singular system is regular; otherwise, the system is singular and the constraint algorithm for linearly singular systems can be applied to obtain the dynamics.

Finally, let us note that there are other equivalent descriptions of the dynamics. For instance, instead of $\hat{\Omega}_L$, we could have used the Euler–Lagrange form on $\mathrm{J}^2\rho$. We omit the details.

6.2 Skinner-Rusk formulation of time-dependent mechanics

A mixed lagrangian-hamiltonian formulation of time-independent mechanics was studied geometrically in a series of papers by Skinner and Rusk [Ski 83, SR 83]. Recently, the time-dependent case has been studied in [CMC 02]. We will show how this can be described in our formalism.

Our starting point is a fibre bundle $\rho\colon Q \to \mathbf{R}$ and a lagrangian function $L\colon \mathrm{J}^1\rho \to \mathbf{R}$. In this formulation, the dynamics is represented by a first-order system on the manifold $M := \mathrm{T}^*Q \times_Q \mathrm{J}^1\rho$. Denote the several projections as in the following diagram:

$$
\begin{array}{c}
\xymatrix{
& & \pi & \\
M := \mathrm{T}^*Q \times_Q \mathrm{J}^1\rho \ar[r]^{\mathrm{pr}_2} \ar[d]_{\mathrm{pr}_1} & \mathrm{J}^1\rho \ar[d]^{\rho_{1,0}} \ar[dr]^{\rho_1} & \\
\mathrm{T}^*Q \ar[r]_{\tau_Q^*} & Q \ar[r]_{\rho} & \mathbf{R}
}
\end{array}
$$

We can define the following function on M:

$$
\mathcal{H} = \langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle - \mathrm{pr}_2^* L,
$$

where $\langle\,,\,\rangle$ denotes the natural pairing between vectors and covectors on Q, and the 2-form on M

$$
\Omega_\mathcal{H} = \mathrm{pr}_1^* \omega_Q - \mathrm{d}\mathcal{H} \wedge \mathrm{d}t,
$$

where ω_Q is the canonical symplectic form on T^*Q; it defines a morphism $\widehat{\Omega}_\mathcal{H}\colon \mathrm{T}M \to \mathrm{T}^*M$.

With these definitions we can write the equations of the dynamics in the Skinner-Rusk formulation, which are

$$
\begin{cases}
i_Z \Omega_\mathcal{H} = 0 \\
i_Z \mathrm{d}t = 1
\end{cases},
$$

for a vector field $Z \in \mathfrak{X}(M)$. These equations are equivalent to the time-dependent linearly singular system on M defined by the following diagram:

$$
\begin{array}{c}
\xymatrix{
\mathrm{J}^1\pi \ar[r]^{(\widehat{\Omega}_\mathcal{H})|_{\mathrm{J}^1\pi}} \ar[d]_{\pi_{1,0}} & \mathrm{T}^*M \\
M \ar[ur]_{\tau_M^*} &
}
\end{array}
$$

7 An example: a simple pendulum of given variable length

Consider a simple pendulum whose length is given by a time-dependent function $R(t)$. Its equation of motion can be written as [GP 92]

$$
\begin{cases}
\dot{x} & = & v_x \\
\dot{y} & = & v_y \\
\dot{v}_x & = & -\tau x \\
\dot{v}_y & = & -\tau y - g \\
x^2 + y^2 & = & R^2(t)
\end{cases},
$$

15

where g is the gravitational acceleration and $\tau R(t)$ is the string tension per unit mass.

This system can be described as a time-dependent linearly singular system in the following way. Take $M := \mathbf{R}^6$, with coordinates $(t, x, y, v_x, v_y, \tau)$, as a configuration manifold fibred over $\mathbf{R}$, with coordinate t. The product $M \times \mathbf{R}^5$ is a trivial vector bundle over M, and the affine bundle morphism $\mathcal{A} \colon \mathrm{J}^1\rho \to M \times \mathbf{R}^5$ defined by

$$\mathcal{A}(\dot{x}, \dot{y}, \dot{v}_x, \dot{v}_y, \dot{\tau})_p = (\dot{x} - v_x, \dot{y} - v_y, \dot{v}_x + \tau x, \dot{v}_y + \tau y + g, -(x^2 + y^2 - R^2(t)))_p,$$

where $p = (t, x, y, v_x, v_y, \tau) \in M$, models the system.

Choosing the connection $\tilde{\Gamma} = \mathrm{d}t \oplus \frac{\partial}{\partial t}$, as described in section 4, we can convert this system into an autonomous linearly singular system on M, which can be written as

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{t} \\ \dot{x} \\ \dot{y} \\ \dot{v}_x \\ \dot{v}_y \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ -\tau x \\ -\tau y - g \\ x^2 + y^2 - R^2(t) \\ 1 \end{pmatrix}$$

We solve this autonomous linearly singular system by means of the constraint algorithm for the autonomous case that is sketched in section 4. In this case, three steps are needed to solve the system. We give here only the constraint functions ϕ^i and manifolds M_i obtained at each step:

1. $\phi^1 = x^2 + y^2 - R^2(t)$,
   $M_1 = \{\phi^1 = 0\}$,
   and the possible solution vector fields are of the form

   $$X \underset{M_1}{\simeq} \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} - \tau x \frac{\partial}{\partial v_x} - (\tau y + g)\frac{\partial}{\partial v_y} + f \frac{\partial}{\partial \tau},$$

   where $f \in \mathrm{C}^\infty(M_1)$ is a function to be determined.

2. $\phi^2 = X \cdot \phi^1 \underset{M_1}{\simeq} x v_x + y v_y - RR'$,
   $M_2 = \{\phi^1 = \phi^2 = 0\}$.

3. $\phi^3 = X \cdot \phi^2 \underset{M_2}{\simeq} v_x^2 + v_y^2 - \tau R^2 - (RR'' + (R')^2)$,
   $M_3 = \{\phi^1 = \phi^2 = \phi^3 = 0\}$.

4. $\phi^4 = X \cdot \phi^3 \underset{M_3}{\simeq} -4\tau RR' - 3v_y g - (RR''' + 3R'R'') - fR^2$.
   The equation $\phi^4 = 0$ determines the function $f \in \mathrm{C}^\infty(M_3)$, so it is not a new constraint and the system is solved.

Using polar coordinates $(t, r, \varphi, v_r, v_\varphi, \tau)$, defined by

$$\begin{cases} x &= r \cos \varphi \\ y &= r \sin \varphi \\ v_x &= v_r \cos \varphi - v_\varphi r \sin \varphi \\ v_y &= v_r \sin \varphi + v_\varphi r \cos \varphi \end{cases},$$

16

we see that the final submanifold M_3 is diffeomorphic to $\mathbf{R} \times \mathrm{T}\mathbf{S}^1$, and it is embedded in M by $(t, \varphi, v_\varphi) \to (t, R(t), \varphi, R'(t), (v_\varphi^2 R(t) - g\sin\varphi - R''(t))/R(t))$. The (unique) solution vector field is

$$X = \frac{\partial}{\partial t} + v_\varphi \frac{\partial}{\partial \varphi} - \frac{2R'(t)v_\varphi + g\cos\varphi}{R(t)} \frac{\partial}{\partial v_\varphi}.$$

Appendix: vector hulls

Vector hulls of affine spaces

Though affine geometry is well-known, it presents some less known canonical structures. The basic fact is that any affine space has a canonical immersion, as a hyperplane, in a vector space; with it, affine maps can be understood as linear maps. We will not go into the details of this construction, nor into its several applications —see for instance [BB 75, Ber 87, GM 04, MMS 02], instead we will describe some of the basic facts.

If A is a real affine space, we denote by $\vec{A}$ its associated vector space. We denote by $\mathcal{A}(A, B)$ the set of affine maps between the affine spaces A and B. For any affine map $f\colon A \to B$ there is an associated linear map $\vec{f} \in \mathcal{L}(\vec{A}, \vec{B})$.

The *vector hull* of A is a vector space $\widehat{A}$ together with an affine map $j\colon A \to \widehat{A}$, and such that the following universal property holds: for every vector space F and affine map $h\colon A \to F$, there exists a unique linear map $\hat{h}\colon \widehat{A} \to F$ such that $h = \hat{h} \circ j$. It turns out that j is an affine immersion, with $j(A)$ a hyperplane in $\widehat{A}$ not containing 0.

$$
\begin{array}{ccc}
A & \xhookrightarrow{\ j\ } & \widehat{A} \\
 & \underset{h}{\searrow} & \big\downarrow{\hat{h}} \\
 & & F
\end{array}
$$

Notice that such a hyperplane can be described as the set $w^{-1}(1)$, with $w\colon \widehat{A} \to \mathbf{R}$ a unique linear form. This also identifies $\vec{A}$ with $w^{-1}(0)$. We can gather all this information in a diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \vec{A} & \xrightarrow{\ i\ } & \widehat{A} & \xrightarrow{\ w\ } & \mathbf{R} & \longrightarrow & 0 \\
 & & & & \big\uparrow{j} & & & & \\
 & & & & A & & & &
\end{array}
$$

From the universal property, it is clear that the assignment $h \mapsto \hat{h}$ is indeed an isomorphism $\mathcal{A}(A, F) \simeq \mathcal{L}(\widehat{A}, F)$, and in particular we have $\mathcal{A}(A, \mathbf{R}) \simeq \widehat{A}^*$.

Given an affine map $f\colon A \to B$, there is a unique linear map $\widehat{f}\colon \widehat{A} \to \widehat{B}$ such that $\widehat{f} \circ j_A = j_B \circ f$. We will call it the *vector extension* of f for obvious reasons.

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
\big\downarrow{j_A} & & \big\downarrow{j_B} \\
\widehat{A} & \xdashrightarrow{\ \widehat{f}\ } & \widehat{B}
\end{array}
$$

Now the assignment $f \mapsto \widehat{f}$ is an affine inclusion $\mathcal{A}(A, B) \hookrightarrow \mathcal{L}(\widehat{A}, \widehat{B})$,

Now let us describe the particular, but important, case where where the affine space is a vector space F. Then we have a canonical identification $\widehat{F} = \mathbf{R} \times F$, with the inclusion $j_F(u) = (1, u)$, and $w_F(\lambda, u) = \lambda$. Given an affine map $h \colon A \to F$, its vector extension $\widehat{h} \colon \widehat{A} \to \widehat{F} = \mathbf{R} \times F$ is given by $\widehat{h} = (w_A, \widehat{h})$.

More particularly, for an affine map $h \colon E \to F$ between vector spaces, $h(u) = h_0 + h_1 \cdot u$, we have that $\widehat{h} \colon E \times \mathbf{R} \to F$ is given by $\widehat{h}(\lambda, u) = h_0 \lambda + h_1 \cdot u$, and the vector extension of h is $\widehat{h}(\lambda, u) = (\lambda, h_0 \lambda + h_1 \cdot u)$.

Finally, let us put coordinates everywhere. Consider a point $e_0 = a_0 \in A$ and a basis $(e_i)_{i \in I}$ of $\vec{A}$. Then, with the appropriate identifications, every point in $\widehat{A}$ can be uniquely written as $x = x^0 e_0 + x^i e_i$. The point x belongs to A iff $x^0 = 1$, and belongs to $\vec{A}$ iff $x^0 = 0$. With these coordinates, the vector extension of an affine map $y^j = c^j + T_i^j x^i$ is the linear map $y^\nu = T_\mu^\nu x^\mu$, with $T_0^j = c^j$, $T_0^0 = 1$, $T_i^0 = 0$.

Vector hulls of affine bundles and jet bundles

All that we have done up to now with affine spaces can be formulated for affine bundles in an analogous way [MMS 02]. In this case, the starting point is an affine bundle $A \to M$ modelled on a vector bundle $\vec{A} \to M$. Without going into technical details, the vector hull of A is the vector bundle $\widehat{A} \to M$ whose fibres are the vector hulls of the fibres of A (which are affine spaces). All the affine and linear maps that we considered before are now affine and vector bundle morphisms over the identity map on M.

For our discussion it is particularly useful the following fact: if we have an exact sequence of vector bundle morphisms $0 \longrightarrow \vec{A} \overset{\alpha}{\longrightarrow} W \overset{w}{\longrightarrow} M \times \mathbf{R} \longrightarrow 0$, then $w^{-1}(1)$ is an affine bundle modelled on $\vec{A}$ (so it is isomorphic to A), and W is canonically isomorphic to the vector hull of $w^{-1}(1)$.

We will apply this to find the vector hulls of the affine bundles that play a role in this paper, that is, the jet bundles $\rho_{1,0} \colon \mathrm{J}^1 \rho \to M$ and $\rho_{2,1} \colon \mathrm{J}^2 \rho \to \mathrm{J}^1 \rho$, when $\rho \colon M \to \mathbf{R}$ is a fibre bundle over the real line. Recall sections 2.1 and 5.1 for the definition and properties of these jet bundles.

From section 2.1 it follows that the sequence of vector bundle morphisms

$$0 \longrightarrow \mathrm{V}\rho \lhook\joinrel\longrightarrow \mathrm{T}M \overset{\mathrm{d}t}{\longrightarrow} M \times \mathbf{R} \longrightarrow 0$$

is exact, where $\mathrm{d}t$ denotes, by abuse of notation, the contraction of a tangent vector with the 1-form $\mathrm{d}t$. Furthermore, it is also seen in section 2.1 that $\mathrm{J}^1 \rho$ can be canonically embedded into $\mathrm{T}M$, and the image of this embedding is just $\mathrm{d}t^{-1}(1)$. Therefore, the vector hull of $\mathrm{J}^1 \rho$ is naturally identified with $\mathrm{T}M$:

$$\widehat{\mathrm{J}^1 \rho} = \mathrm{T}M.$$

Now we consider the bundle $\rho_{2,1} \colon \mathrm{J}^2 \rho \to \mathrm{J}^1 \rho$. We have seen in section 5.1 that it is an affine bundle modelled on the vector bundle $\mathrm{V}\rho_{1,0}$, which is a subbundle of the tangent bundle $\mathrm{T}(\mathrm{J}^1 \rho)$, and that $\mathrm{J}^2 \rho$ can be naturally embedded into $\mathrm{T}(\mathrm{J}^1 \rho)$. Our aim is to identify the vector hull $\widehat{\mathrm{J}^2 \rho}$ with a suitable subbundle of $\mathrm{T}(\mathrm{J}^1 \rho)$.

The Cartan distribution $C\rho_{1,0}$ is a subbundle of $\mathrm{T}(\mathrm{J}^1\rho)$ which includes $\mathrm{V}\rho_{1,0}$, and it is straightforward to see that the sequence of vector bundle morphisms

$$0 \longrightarrow \mathrm{V}\rho_{1,0} \lhook\joinrel\longrightarrow C\rho_{1,0} \overset{w}{\longrightarrow} \mathrm{J}^1\rho \times \mathbf{R} \longrightarrow 0$$

is exact, where w denotes denotes the contraction with the 1-form $\mathrm{d}t$, restricted to vectors in $C\rho_{1,0}$. It is easy to see that $w^{-1}(1)$ is equal to $\mathrm{J}^2\rho$ as a submanifold of $\mathrm{T}(\mathrm{J}^1\rho)$, so we can conclude that $\widehat{\mathrm{J}^2\rho}$ is naturally identified with $C\rho_{1,0}$:

$$\widehat{\mathrm{J}^2\rho} = C\rho_{1,0}.$$

Acknowledgments

References

[AM 78]   R. ABRAHAM AND J. E. MARSDEN, *Foundations of Mechanics* (2nd ed.), Addison–Wesley, Reading, 1978.

[BB 75]   Y. BAMBERGER ET J.-P. BOURGUIGNON, "Torseurs sur un espace affine", pp. 151–202 in L. SCHWARTZ, *Les tenseurs*, Hermann, Paris, 1975.

[Ber 87]   M. BERGER, *Geometry I*, Springer-Verlag, Berlin, 1987 (translated from the original French edition of 1977).

[Car 90]   J. F. CARIÑENA, "Theory of singular lagrangians", *Fortschr. Phys.* **38** (1990) 641–679.

[CF 93]   J. F. CARIÑENA AND J. FERNÁNDEZ-NÚÑEZ, "Geometric theory of time-dependent singular lagrangians", *Fortschr. Phys.* **41** (1993) 517–552.

[CLM 94]   D. CHINEA, M. DE LEÓN AND J. C. MARRERO, "The constraint algorithm for time-dependent lagrangians", *J. Math. Phys.* **35** (1994) 3410–3447.

[CMC 02]   J. CORTÉS, S. MARTÍNEZ AND F. CANTRIJN, "Skinner-Rusk approach to time-dependent mechanics", *Phys. Lett. A* **300** (2002) 250–258.

[Del 04]   M. DELGADO, *Métodos geométricos en problemas de control óptimo singulares: fundamentos y aplicaciones*, Ph. D. thesis, Univ. Carlos III de Madrid, 2004.

[EMR 91]   A. ECHEVERRÍA ENRÍQUEZ, M. C. MUÑOZ LECANDA AND N. ROMÁN ROY, "Geometrical setting of time-dependent regular systems. Alternative models", *Rev. Math. Phys.* **3** (1991) 301–330.

[GNH 78]   M. J. GOTAY, J. M. NESTER AND G. HINDS, "Presymplectic manifolds and the Dirac-Bergmann theory of constraints", *J. Math. Phys.* **19** (1978) 2388–2399.

[GM 04]   X. GRÀCIA AND R. MARTÍN, "Vector hulls of affine spaces and affine bundles", in preparation.

[GMR 04]   X. GRÀCIA, M. C. MUÑOZ-LECANDA AND N. ROMÁN-ROY, "On some aspects of the geometry of differential equations in physics", *Int. J. Geometric Methods Mod. Phys.* **1** (2004) 265–284.

[GP 91]   X. Gràcia and J. M. Pons, "Constrained systems: a unified geometric approach", *Int. J. Theor. Phys.* **30** (1991) 511–516.

[GP 92]   X. Gràcia and J. M. Pons, "A generalized geometric framework for constrained systems", *Diff. Geom. Appl.* **2** (1992) 223–247.

[GP 02]   X. Gràcia and J. M. Pons, "Symmetries and infinitesimal symmetries of singular differential equations", *J. Phys. A: Math. Gen.* **35** (2002) 5059–5077.

[ILMM 99] A. Ibort, M. de León, J. C. Marrero and D. Martín de Diego, "Dirac brackets in constrained dynamics", *Fortsch. Phys.* **47** (1999) 459–492.

[Kru 97]   O. Krupková, *The geometry of ordinary variational equations*, LNM 1678, Springer, Berlin, 1997.

[LMM 96]   M. de León, J. Marín-Solano and J. C. Marrero, "The constraint algorithm in the jet formalism", *Diff. Geom. Appl.* **6** (1996) 275–300.

[LMMMR 02] M. de León, J. Marín-Solano, J. C. Marrero, M. C. Muñoz-Lecanda and N. Román-Roy, "Singular lagrangian systems on jet bundles", *Fortsch. Phys.* **50** (2002) 105–169.

[MMT 95]   G. Mendella, G. Marmo and W. M. Tulczijew, "Integrability of implicit differential equations", *J. Phys. A: Math. Gen.* **28** (1995) 149–163.

[MMS 02]   E. Martínez, T. Mestdag and W. Sarlet, "Lie algebroid structures and lagrangian systems on affine bundles", *J. Geom. Phys.* **44** (2002) 70–95.

[MPL 03]   E. Massa, E. Pagani and S. Vignolo, "Legendre transformation and analytical mechanics: a geometric approach", *J. Math. Phys.* **44** (2003) 1709–1722.

[MS 98]   L. Mangiarotti and G. Sardanashvily, *Gauge mechanics*, World Scientific, Singapore, 1998.

[Olv 93]   P. J. Olver, *Applications of Lie groups to differential equations*, 2nd ed., GTM 107, Springer, New York, 1993.

[RR 94]   P. J. Rabier and W. C. Rheinboldt, "A geometric treatment of implicit differential-algebraic equations", *J. Differential Equations* **109** (1994) 110–146.

[Sau 89]   D. J. Saunders, *The Geometry of Jet Bundles*, London Mat. Soc. Lecture Note Series 142, Cambridge University Press, Cambridge, 1989.

[Ski 83]   R. Skinner, "First-order equations of motion for classical mechanics", *J. Math. Phys.* **24** (1983) 2581–2588.

[SR 83]   R. Skinner and R. Rusk, "Generalized hamiltonian dynamics. I. Formulation on $T^*Q \oplus TQ$", *J. Math. Phys.* **24** (1983) 2589–2594.

[Vig 00]   S. Vignolo, "A new presymplectic geometrical framework for time-dependent lagrangian systems: the constraint algorithm and the second-order differential equation problem", *J. Phys. A: Math. Gen.* **33** (2000) 5117–5135.